1. Fontaine's period ring formalism.

We want to study continuous representations of $G_K = \mathrm{Gal}(K^s/K)$ for a p-adic local field K (mainly a finite extension of $\mathbb{Q}_p$, but also can be a local function field) on a finitely generated $\mathbb{F}_p$-vector space/$\mathbb{Z}_p$-module/$\mathbb{Q}_p$-vector space.

A general strategy of Fontaine is to construct an interesting ring with interesting structures and Galois action and we can study the representation by base-changing to the ring. More precisely, let's say G is a group, L is a field, and we want to study representations of G on a finite-dimensional L-vector space. Then what we generally want to do is to find an L-algebra B, which is a domain, with a L-linear G-action. Then, we can endow the diagonal G-action on $V \otimes_L B$, and define $D_B(V) = (V \otimes_L B)^G$, which is a B^G-module. From this, we have a G-equivariant map

$$D_B(V) \otimes_{B^G} B \longrightarrow V \otimes_L B.$$

We want good situations where this map is isomorphism. Then, we can hope to use structures of B (and $D_B(V)$) to study V.

Definition 1.1. *The L-algebra B is called G-**regular** if*

- $B^G = (\mathrm{Frac}\, B)^G$,
- *if $0 \neq b \in B$ and $L \cdot b$ is a G-stable line, then $b \in B^\times$.*

Obviously, if B is G-regular, then $E = B^G$ is a field.

Proposition 1.1. *If B is G-regular, $\dim_E D_B(V) \leq \dim_L V$, and the map $D_B(V) \otimes_{B^G} B \longrightarrow V \otimes_L B$ is injective.*

Definition 1.2. *For a G-regular ring B, if $\dim_E D_B(V) = \dim_L V$ (or equivalently the map $D_B(V) \otimes_{B^G} B \longrightarrow V \otimes_L B$ is an isomorphism), then we say V is B-**admissible**.*

Remark 1.1.
(1) Let ρ denote the G-representation on V, and let $n = \dim_L V$. Then, upon choosing a basis of V, we can see ρ as a cocycle $\rho \in H^1(G, \mathrm{GL}_n(L))$, where $\mathrm{GL}_n(L)$ has a trivial G-action. Then, saying (ρ, V) is B-admissible is equivalent to saying that

$$\rho \in \ker(H^1(G, \mathrm{GL}_n(L)) \longrightarrow H^1(G, \mathrm{GL}_n(B))).$$

(2) If G is a topological group and B is a topological L-algebra with continuous G-action, the above remark holds for continuous representations (ρ, V) and continuous H^1.
(3) In practice, B comes with additional structures (e.g. endomorphism, filtration, $\cdots$) commuting with G-action. Given V, these induce additional structures on $D_B(V)$. Our general hope is that, by putting enough additional structures, we can recover B-admissible V from $D_B(V)$.

2. φ-modules.

We study one of the easiest examples of the above situation. Let F be a local field of characteristic p, of form $\mathbb{F}_q((t))$, $G = G_F$ and $L = \mathbb{F}_p$.

In this case, let $B = F^s$. There is obviously an action of G_F, and as it is a field, it is obviously G_F-regular, and $B^{G_F} = F$.

Lemma 2.1. *Every continuous G_F-representation on a finite-dimensional $\mathbb{F}_p$-vector space is F^s-admissible.*

Proof. This is immediate from Hilbert 90 (so that $H^1(G_F, \mathrm{GL}_n(F^s)) = 0$) and the cohomological way of seeing B-admissibility. $\square$

Now a nice thing is that we have a Frobenius endomorphism $\varphi : F^s \longrightarrow F^s$, $x \mapsto x^p$, which commutes with G_F-action. Thus, given V, we get an induced map $\phi : D(V) \xrightarrow{1\otimes\varphi} D(V)$ where $D(V) = (V \otimes_{\mathbb{F}_p} F^s)^{G_F}$.

Definition 2.1. *A φ-**module** over a ring A with an endomorphism $\varphi : A \longrightarrow A$ is a finitely generated A-module D, together with a φ-semilinear map $\phi : D \longrightarrow D$, such that the A-linear map $\varphi^* D = D \otimes_{A,\varphi} A \xrightarrow{d\otimes 1 \mapsto \varphi(d)} D$ is an isomorphism.*

Thus, $D(V)$ is a φ-module over F, because φ is injective so that ϕ on $D(V)$ is injective and thus the linearization of ϕ is injective too.

Corollary 2.1. *The functor*

$$D : \left\{ \begin{array}{c} \text{continuous representations of} \\ G_F \text{ on finite-dimensional} \\ \mathbb{F}_p\text{-vector spaces} \end{array} \right\} \longrightarrow \{\varphi\text{-modules over } F\},$$

is fully faithful.

Theorem 2.1 (Fontaine)**.** *The above functor D is an equivalence of categories, with quasi-inverse*

$$D \mapsto \mathbf{V}(D) := (D \otimes_F F^s)^{\varphi = \mathrm{id}},$$

where $\varphi : D \otimes_F F^s \longrightarrow D \otimes_F F^s$ is φ diagonally.

Proof. It is fully faithful, as this gives a quasi-inverse to the essential image: given a representation V,

$$D(V) \otimes_F F^s \xrightarrow{\sim} V \otimes_{\mathbb{F}_p} F^s,$$

is not only Galois equivariant but φ-equivariant, where φ on the RHS is just φ on the second factor, and $(F^s)^{\varphi=\mathrm{id}} = \mathbb{F}_p$.

For essential surjectivity, it reduces to show that we don't lose dimension via $\mathbf{V}$. Namely, given a φ-module F, we have to show $\dim_{\mathbb{F}_p} \mathbf{V}(D) = \dim_F D$. One uses F^s is separably closed and counts roots. $\square$

Remark 2.1. It might be hard to construct Galois representations as the structure of Galois group is mysterious. Thus, this theory can be thought as an easy explicit way of producing Galois representations of G_F.

Now we want to "lift" the situation to torsion coefficients and $\mathbb{Q}_p$-coefficients. To do this, we "lift" F^s to characteristic 0.

Definition 2.2. *Let $(\mathcal{O}_{\mathcal{E}}, \varphi)$ be a Cohen ring for F, i.e. $\mathcal{O}_{\mathcal{E}}$ is a complete dvr with uniformizer p with residue field F, and φ is a lift of Frobenius on F.*

Example 2.1.
(1) If F is perfect, then $(W(F), \varphi)$ is the unique possible Cohen ring, where φ is the unique lift of Frobenius.
(2) In our case of $F = \mathbb{F}_q((t))$, we can take $\mathcal{O}_{\mathcal{E}} = W(\mathbb{F}_q)((t))^\wedge$, where $\wedge$ means we take p-adic completion. One can take φ to be the lift of Frobenius on $W(\mathbb{F}_q)$ (which we don't have any choice) and $t \mapsto t^p$ (which we have freedom to choose; we might as well take $t \mapsto (1+t)^p - 1$, which turns out to be more useful).

Definition 2.3. *We define as follows.*

- $\mathcal{E} = \mathcal{O}_{\mathcal{E}}[1/p]$.
- $\mathcal{E}^{\mathrm{un}}$ *is the maximal unramified extension of* $\mathcal{E}$.
- $\mathcal{O}_{\mathcal{E}^{\mathrm{un}}}$ *is the ring of integers of* $\mathcal{E}^{\mathrm{un}}$.
- $\widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}}$ *is the* p*-adic completion of* $\mathcal{O}_{\mathcal{E}^{\mathrm{un}}}$.
- $\widehat{\mathcal{E}}^{\mathrm{un}} = \widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}}[1/p]$.

The extension $\mathcal{E}^{\mathrm{un}}/\mathcal{E}$ *is Galois with Galois group* G_F*, and we have* φ *and* G_F*-actions on all the above rings.*

As we have seen before, it might be useful to calculate G_F and φ-invariants of the above rings.

Lemma 2.2. *(1)* $\mathcal{O}_{\mathcal{E}} \xrightarrow{\sim} (\widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}})^{G_F}$.

(2) $\mathbb{Z}_p = (\widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}})^{\varphi=\mathrm{id}}$.

(3) $H^1(G_F, \mathrm{GL}_n(\widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}})) = 1$.

This allows us to lift to characteristic zero.

Proof. We use successive approximation, by noticing that we can filter $\widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}}$ ($\mathrm{GL}_n(\widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}})$, resp.) such that graded pieces are F^s (either $\mathrm{GL}_n(F^s)$ or $M_n(F^s)$, resp.). Note that for GL_n we use obvious congruence subgroups. And we know analogous calculations for graded pieces. □

Corollary 2.2. *(1) Let* Λ *be a finitely generated* $\mathbb{Z}_p$*-module with continuous* $\mathbb{Z}_p$*-linear* G_F*-action. Then,*

$$D(\Lambda) := (\Lambda \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}})^{G_F},$$

is a φ*-module over* $\mathcal{O}_{\mathcal{E}}$*, and*

$$D(\Lambda) \otimes_{\mathcal{O}_{\mathcal{E}}} \widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}} \longrightarrow \Lambda \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}},$$

is a (G_F, φ)*-equivariant isomorphism.*

(2) This functor gives an equivalence of categories

$$\left\{\begin{matrix}\text{continuous } G_F\text{-representations}\\ \text{on finitely generated}\\ \mathbb{Z}_p\text{-modules}\end{matrix}\right\} \longrightarrow \{\varphi\text{-modules over } \mathcal{O}_{\mathcal{E}}\},$$

with quasi-inverse

$$\mathbf{V}(D) := (D \otimes_{\mathcal{O}_{\mathcal{E}}} \widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}})^{\varphi=\mathrm{id}}.$$

(3) Now let V *be a finite-dimensional* G_F*-representation over* $\mathbb{Q}_p$*. Then,*

$$D(V) := (V \otimes_{\mathbb{Q}_p} \widehat{\mathcal{E}}^{\mathrm{un}})^{G_F},$$

is a fully faithful functor

$$\left\{\begin{matrix}\text{continuous } G_F\text{-representations}\\ \text{on finite dimensional}\\ \mathbb{Q}_p\text{-vector spaces}\end{matrix}\right\} \longrightarrow \{\varphi\text{-modules over } \mathcal{E}\},$$

with essential image being the subcategory of ***étale*** φ***-modules****. Here,* (D, φ)*, a* φ*-module over* $\mathcal{E}$*, is an étale* φ*-module, if there is a* φ*-module* (D', φ') *on* $\mathcal{O}_{\mathcal{E}}$ *such that* $(D, \varphi) \cong (D', \varphi') \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{E}$.

Proof. Everything is almost a formal consequence of characteristic p coefficient theory. □

Remark 2.2. That we need to introduce the notion of étale φ-modules is very much expected. Indeed, if φ acts on a φ-module over $\mathcal{E}$ with an eigenvalue, say, p, then obviously it does not come from φ-module over $\mathcal{O}_{\mathcal{E}}$. In some sense being étale φ-module is to assert that there is an "integral model." On the other hand, for any rational Galois representation, there is a Galois stable lattice by usual compactness argument.

3. (φ, Γ)-**modules.**

Characteristic p theory is very satisfying. But what about mixed characteristic local fields? Let $K/\mathbb{Q}_p$ be a finite extension, then how do we describe continuous G_K-reprsentations on finitely generated $\mathbb{F}_p$-vector spaces/$\mathbb{Z}_p$-modules/$\mathbb{Q}_p$-representaitons?

The idea is to find a (deeply ramified, infinite) Galois extension K_∞/K such that

- one can transfer the situation to equicharacteristic local fields, i.e. there is an equicharacteristic local field F such that $G_{K_\infty} \cong G_F$,
- the transferred continuous action of G_{K_∞} on $\widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}}$ extends to a continuous G_K-action on $\widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}}$ commuting with φ,
- and hopefully $\Gamma = \mathrm{Gal}(K_\infty/K)$ is as simple as possible.

If this is the case, then we get a continuous Γ-action on $(\widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}})^{G_{K_\infty}} = \mathcal{O}_{\mathcal{E}}$ commuting with φ.

Definition 3.1. *If the above situation holds, a (φ, Γ)-**module** over $\mathcal{O}_{\mathcal{E}}$ ($\mathcal{E}$, resp.) is a φ-module over $\mathcal{O}_{\mathcal{E}}$ ($\mathcal{E}$, resp.) with a semilinear Γ-action commuting with φ.*

*A (φ, Γ)-module over $\mathcal{E}$ is **étale** if the underlying φ-module is étale.*

Remark 3.1. The definition of étale (φ, Γ)-module is sensible, as Γ is compact.

Theorem 3.1. *(1) There is an equivalence of categories*

$$\left\{\begin{matrix}\text{continuous} \\ G_K\text{-representations on finitely} \\ \text{generated } \mathbb{Z}_p\text{-modules}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}(\varphi, \Gamma)\text{-modules over} \\ \mathcal{O}_{\mathcal{E}}\end{matrix}\right\},$$

$$\Lambda \longmapsto (\Lambda \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}})^{G_{K_\infty}},$$

with quasi-inverse

$$D \longmapsto (D \otimes_{\mathcal{O}_{\mathcal{E}}} \widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}})^{\varphi=\mathrm{id}},$$

where G_K acts on the target diagonally (G_K acts via its quotient Γ on D).

(2) There is an equivalence of categories

$$\left\{\begin{matrix}\text{continuous} \\ G_K\text{-representations on finite} \\ \text{dimensional } \mathbb{Q}_p\text{-modules}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{étale } (\varphi, \Gamma)\text{-modules} \\ \text{over } \mathcal{E}\end{matrix}\right\},$$

$$V \longmapsto (V \otimes_{\mathbb{Q}_p} \widehat{\mathcal{E}}^{\mathrm{un}})^{G_{K_\infty}},$$

with quasi-inverse

$$D \longmapsto (D \otimes_{\mathcal{E}} \widehat{\mathcal{E}}^{\mathrm{un}})^{\varphi=\mathrm{id}}.$$

Proof. Everything is a very formal consequence. ☐

4. **Tilting equivalence.**

Characteristic 0 situation can be salvaged as a formal consequence of having K_∞, but is there such K_∞? Classically, Fontaine and Wintenberger used "norm fields" to find K_∞. Nowadays we have a more conceptual perspective due to Scholze using tilting equivalence of perfectoid fields.

Definition 4.1. *A* ***perfectoid field*** *K is a complete nonarchimedean field of residue charateristic p, complete with respect to a valuation v_k (resp. norm $|-|$) such that*

- *v_k is non-discrete, i.e. the maximal ideal $\mathfrak{m} \subset \mathcal{O}_K$ satisfies $\mathfrak{m}^2 = \mathfrak{m}$,*
- *$\mathrm{Frob} : \mathcal{O}_K/p \longrightarrow \mathcal{O}_K/p$, $x \mapsto x^p$, is surjective.*

Example 4.1. The following are examples of perfectoid fields.

(1) $\mathbb{F}_q((X^{1/p^\infty})) := \left(\cup_{n\geq 0}\mathbb{F}_q((X^{1/p^n}))\right)^\wedge$, the X-adic completion.
(2) $\overline{\mathbb{F}_q((X))}^\wedge$.
(3) $\overline{\mathbb{Q}}_p^\wedge$, the p-adic completion.
(4) Given a finite extension $F/\mathbb{Q}_p$, $F(\pi^{1/p^\infty})^\wedge$, for a uniformizer $\pi \in F$.
(5) Given a finite extension $F/\mathbb{Q}_p$, and a compatible system (ε_n) of p-power primitive roots of 1, $F(\varepsilon_n, n \geq 1)^\wedge$.

Remark 4.1. If K is a complete nonarchimedean field of characteristic p with non-discrete valuation, then K is perfectoid if and only if K is perfect.

Definition 4.2. *Let K be a perfectoid field. Fix a pseudo-uniformizer $\varpi \in \mathcal{O}_K$, namely $|p| \leq |\varpi| < 1$. Define*

$$\mathcal{O}_{K^\flat} := \varprojlim_{\mathrm{Frob}} \mathcal{O}_K/\varpi.$$

Choose a compatible system of p-power roots of ϖ, $\varpi^\flat = (0, \varpi_1^\flat, \cdots) \in \mathcal{O}_{K^\flat}$, such that $\varpi_1^\flat \neq 0$. Then, we define the ***tilt*** *of K to be*

$$K^\flat = \mathcal{O}_{K^\flat}[1/\varpi^\flat].$$

Lemma 4.1. *Fix a valuation v_K on a perfectoid field K.*

(1) $\mathcal{O}_{K^\flat}$ has a valuation defined by

$$(x_0, x_1, \cdots) \mapsto \lim_{n\to\infty} v_K(\tilde{x}_n^{p^n}),$$

where $\tilde{x}_n \in \mathcal{O}_K$ is a lift of x_n. Namely, the limit always exists, and does not depend on a choice of lifts.

(2) $\mathcal{O}_{K^\flat}$ is complete with respect to this valuation, and $\mathcal{O}_{K^\flat}$ does not depend on choices of ϖ. The topology defined by this valuation is independent of choice of v_K.

(3) $K^\flat$ is a perfectoid field of characteristic p, and does not depend on choice of $\varpi^\flat$.

Remark 4.2. If K is a perfectoid field of characteristic p, then $K^\flat \cong K$. More generally, if K is characteristic 0,

$$\mathcal{O}_{K^\flat} = \varprojlim_{\mathrm{Frob}} \mathcal{O}_K/\varpi \xleftarrow{\sim} \varprojlim_{x\mapsto x^p} \mathcal{O}_K,$$

because the rightmost object is a priori just a multiplicative monoid but $x \mapsto x^p$ commutes with addition in characteristic p. In general, if $(x^{(n)}), (y^{(n)})$ is in $\varprojlim_{x\mapsto x^p} \mathcal{O}_K$, then we can give the additive structure by

$$(x^{(n)}) + (y^{(n)}) = \left(\lim_{m\to\infty}\left(x^{(n+m)} + y^{(n+m)}\right)^{p^m}\right).$$

Theorem 4.1 (Scholze)**.** *Let K be a perefctoid field.*

(1) If L/K is a finite extension, then L is perfectoid.

*(2) **(Tilting equivalence)** There is a degree-preserving equivalence of categories*

$$\{\textit{finite extensions of } K\} \longrightarrow \left\{\textit{finite extension of } K^\flat\right\},$$

$$L \longmapsto L^\flat.$$

In particular, we have a canonical isomorphism $G_K \xrightarrow{\sim} G_{K^\flat}$.

Remark 4.3. (1) One can give an explicit quasi-inverse. Namely, given K there is a **canonical ring homomotphism**

$$\theta_K : W(\mathcal{O}_{K^\flat}) \twoheadrightarrow \mathcal{O}_K,$$

which is defined on Teichmuller representatives as

$$[(x_0, x_1, \cdots)] \longmapsto \lim_{n\to\infty} \tilde{x}_n^{p^n},$$

where $\tilde{x}_n \in \mathcal{O}_K$ is a lift of x_n. Then, given a finite extension $E/k^\flat$ with the ring of integers $\mathcal{O}_E$,

$$L = W(\mathcal{O}_E) \otimes_{W(\mathcal{O}_{K^\flat}),\theta_K} K,$$

is an untilt of E.

(2) In fact, we have a more general tilting equivalence,

$$\{\text{perfectoid } K\text{-algebras}\} \xrightarrow{\sim} \left\{\text{perfectoid } K^\flat\text{-algebras}\right\},$$

$$R \longmapsto R^\flat,$$

where a **perfectoid K-algebra** is a Banach K-algebra R such that
- $R^\circ \subset R$ is open and bounded, where R° is the set of power-bounded elements,
- and Frob : $R^\circ/\varpi \longrightarrow R^\circ/\varpi$ is surjective.

More generally, there is the **almost purity theorem**: given a perfectoid K-algebra R, tilting induces an equivalence

$$\{\text{finite étale } R\text{-algebras}\} \leftrightarrow \left\{\text{finite étale } R^\flat\text{-algebras}\right\}.$$

(3) Without fixing a base field K, tilting is not an equivalence: there are many different fields giving the same tilt. For example, we can take $K_1 = \mathbb{Q}_p(\varepsilon_n, n \geq 1)^\wedge$, where ε_n is the compatible system of p-power roots of 1, and $K_2 = \mathbb{Q}_p(p^{1/p^\infty})^\wedge$. Then, $(K_1)^\flat \cong (K_2)^\flat \cong (K_3)^\flat = K_3$, where $K_3 = \mathbb{F}_p((X^{1/p^\infty}))$. For example, $\mathbb{F}_p((X)) \xrightarrow{X \mapsto \varpi^\flat} K_2^\flat$, for $\varpi^\flat = (p, p^{1/p}, \cdots)$, induces an isomorphism $K_3 \xrightarrow{\sim} K_2^\flat$.

Idea of proof of tilting equivalence. We fix $K, \mathcal{O}_K, \varpi, K^\flat, \mathcal{O}_{K^\flat}, \varpi^\flat$. Then, we have an isomorphism $\mathcal{O}_K/\varpi \cong \mathcal{O}_{K^\flat}/\varpi^\flat$.

We prove that $L \mapsto L^\flat$ is an equivalence as follows. We eventually prove that $\mathcal{O}_L \mapsto \mathcal{O}_{L^\flat}$ is an equivalence. Given $\mathcal{O}_L$, we show that $\mathcal{O}_L/\varpi^n$ is the unique flat lift of $\mathcal{O}_L/\varpi$ over $\mathcal{O}_K/\varpi^n$. This is shown by the naturality of cotangent complexes and perfectoidness assumption, the cotangent complexes have to vanish. Thus from the isomorphism $\mathcal{O}_K/\varpi \cong \mathcal{O}_{K^\flat}/\varpi^\flat$, the equivalence follows. □

5. **Back to (φ, Γ)-modules.**

Now using the tilitng equivlance, there is a hope to find "K_∞". Let $K/\mathbb{Q}_p$ be a finite extension, and $F/\mathbb{F}_p((X))$ be also a finite extension. Let $C_1 = \mathbb{C}_p = \widehat{\overline{K}}$, and $C_2 = \widehat{F^s}$. Then, the G_K-action on $\overline{K}$ extends to a continuous action of G_K on C_1, and the G_F-action on $\overline{F}$ extends to a continuous action of G_F on C_2.

Lemma 5.1. *(1) (Krasner's lemma) C_1 and C_2 are algebraically closed. In particular, $C_2 = \widehat{\overline{F}}$.*

(2) C_1 and C_2 are perfectoid, and $C_2 \cong C_1^\flat$. On the other hand, the isomorphism depends on the choice of $\varpi^\flat \in C_1^\flat$, which is highly non-unique.

(3) (Ax-Sen lemma)Let $H \subset G_K$ (resp. $H \subset G_F$) be a closed subgroup, then $C_1^H = (\overline{K}^H)^\wedge$ (resp. $C_2^H = (((F^s)^H)^{\mathrm{perf}})^\wedge$; need to deal with inseparable extensions).

Now fix $\varepsilon_n \in \overline{K}$ a compatible sequence of p-power roots of unity. Our sought-after K_∞ can now be taken as follows.

Definition 5.1. *Let K_∞ be defined as*

$$K_\infty = K(\varepsilon_n, n \geq 1).$$

We see that K_∞ satisfies the following properties.

- K_∞/K is a Galois extension.
- The Galois group $\Gamma = \mathrm{Gal}(K_\infty/K)$ can be realized as an open subgroup of $\mathbb{Z}_p^\times$ via the **cyclotomic character**

$$\chi : \Gamma \longrightarrow \mathbb{Z}_p^\times,$$
$$g \cdot \varepsilon_n = \varepsilon_n^{\chi(g)} \text{ for all } n.$$

  For example, if $K = \mathbb{Q}_p$, then $\Gamma = \mathbb{Z}_p^\times$.
- $\widehat{K_\infty}$ is a perfectoid field.
- Completion does not harm Galois group, namely we have an equivalence of categories

$$\{\text{finite separable extensions of } K_\infty\} \xrightarrow{\sim} \left\{\text{finite separable extensions of } \widehat{K_\infty}\right\},$$
$$L \mapsto \hat{L}.$$

  Thus, via the tilting equivalence, we have an equivalence of categories

$$\{\text{finite separable extensions of } K_\infty\} \xrightarrow{\sim} \left\{\text{finite separable extensions of } \widehat{K_\infty}^\flat\right\},$$
$$L \mapsto \hat{L}^\flat.$$

- Let $\mathbb{F}_q$ be the residue field of K_∞.
  - If $K/\mathbb{Q}_p$ is unramified, then there is a natural map

$$\mathbb{F}_q((X)) \longrightarrow K_\infty^\flat,$$
$$X \mapsto (\varepsilon_0, \varepsilon_1, \cdots) - 1.$$

    This induces the isomorphism $\mathbb{F}_q((X^{1/p^\infty})) \xrightarrow{\sim} \widehat{K}_\infty^\flat$; you can check this by hand.
  - Indeed, it is enough to show that $\bigcup_{n\geq 0} \mathbb{F}_q[[X^{1/p^n}]] \subset \mathcal{O}_{\widehat{K}_\infty^\flat}$ is dense. As $\mathcal{O}_{\widehat{K}_\infty^\flat} = \varprojlim_\varphi \mathcal{O}_{K_\infty}/p$, you need to show that $\overline{\pi}_n \in \mathrm{pr}_m(\bigcup_j \mathbb{F}_q[[X^{1/p^j}]])$, where $\mathrm{pr}_m : \varprojlim_\varphi \mathcal{O}_{K_\infty}/p \longrightarrow \mathcal{O}_{K_\infty}/p = \bigcup_{n\geq 0} W(\mathbb{F}_q)[\pi_n]/p$ is the m-th projection map and $\overline{\pi}_n$ is the class of π_n. But, $\mathrm{pr}_m(X^{p^{m-n}}) = \overline{\pi}_n$ for all n.

- More generally, if $K/\mathbb{Q}_p$ is a general finite extension, then using the above result for L, the maximal unramified subextension of $K/\mathbb{Q}_p$, we get the similar result.

Thus, by the same reason as above an equivalence of categories

$$\left\{\text{finite separable extensions of } \mathbb{F}_q((X))^{\mathrm{perf}}\right\} \xrightarrow{\sim} \left\{\text{finite separable extensions of } \widehat{K_\infty}^{\flat}\right\},$$

$$L \mapsto \hat{L}.$$

- Finally, as the étale topology does not see inseparable extensions, we have an equivalence of categories

$$\left\{\text{finite separable extensions of } \mathbb{F}_q((X))\right\} \xrightarrow{\sim} \left\{\text{finite separable extensions of } \mathbb{F}_q((X))^{\mathrm{perf}}\right\},$$

$$L \mapsto L^{\mathrm{perf}}.$$

This gives the half of the desired properties for K_∞:

Proposition 5.1. *For $F = \mathbb{F}_q((X))$, $G_{K_\infty} \cong G_F$.*

What about the other half?

- We note that G_K acts on $\mathbb{C}_p$, $\mathbb{C}_p^\flat \supset \widehat{K}_\infty^\flat \supset F$, $\mathbb{C}_p^\flat \supset F^s$.
- Thus, G_K acts on $W(\mathbb{C}_p^\flat)$ and $W(\mathcal{O}_{\mathbb{C}_p^\flat})$.
- The G_K action on $\mathbb{C}_p^\flat$ preserves F^s, and the restriction of G_K-action to F^s induces the canonical $G_F \cong G_{K_\infty} \subset G_K$ on F^s.
- Let $\mathcal{O}_{\mathcal{E}}$ be our Cohen ring, $\mathcal{O}_{\mathcal{E}} = W(\mathbb{F}_q)((X))^\wedge$. Indeed, this embeds into $W(\mathbb{C}_p^\flat)$, with Γ and φ-stable image.
  - If $K/\mathbb{Q}_p$ is unramified, the map can be given via

$$X \mapsto [(\varepsilon_0, \varepsilon_1, \cdots)] - 1.$$

    The reason why we take Teichmuller lift is because we cannot take any lift, as we want X to be topologically nilpotent on the LHS.
  - The reason why the image is Γ and φ-stable is because you explicitly know how Γ and Frobenius act on this thing, namely $\gamma.X = (1+X)^{\chi(\gamma)} - 1 \in W(k)((X))^\wedge$ and $\varphi(X) = (1+X)^p - 1 \in W(k)((X))^\wedge$.
  - For $K/\mathbb{Q}_p$ a general finite extension, one uses the maximal unramified subextension and draw the same conclusion (but with less explicit map).
- $\widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}}$, the p-adic completion of the maximal unramified extension of $\mathcal{O}_{\mathcal{E}}$ in $W(\mathbb{C}_p^\flat)$, is stable under G_K and Frobenius, and the restriction of G_K-action on $\widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}}$ to $G_F \cong G_{K_\infty} \subset G_K$ is the canonical action.

Thus we have the desired other half. Namely, we can use this specific K_∞ for the abstract theory of (φ, Γ)-modules.

Theorem 5.1. *There are bijections*

$$\left\{\begin{array}{c}\text{continuous}\\ \text{representations of}\\ G_K \text{ on finitely}\\ \text{generated}\\ \mathbb{Z}_p\text{-modules}\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{c}(\varphi,\Gamma)\text{-modules}\\ \text{over } \mathcal{O}_{\mathcal{E}}\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{c}(\varphi, G_K)\text{-}\\ \text{modules over}\\ \widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}}\end{array}\right\} = \left\{\begin{array}{c}\varphi\text{-module over } \widehat{\mathcal{O}}_{\mathcal{E}^{\mathrm{un}}}\\ \text{with continuous}\\ \text{semilinear}\\ G_K\text{-action}\\ \text{commuting with } \varphi\end{array}\right\}.$$

6. φ-modules and the Fargues-Fontaine curve.

Let F be a perfectoid field of characteristic p.

Definition 6.1. *Let $A_{\mathrm{inf}} = A_{\mathrm{inf}}(F) := W(\mathcal{O}_F)$, equipped with Frobenius.*

Choose a pseudouniformizer $\varpi \in \mathcal{O}_F$.

Definition 6.2. *Let*

$$Y_F = \mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}}) \backslash V(p[\varpi]),$$

an adic space, independent of choice of ϖ, equipped with an action φ such that $\varphi^{\mathbb{Z}}$ acts totally discontinuously and freely.

We will try to justify this construction for a while.

- As a set, $\mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}})$ is the set of continuous multiplicative valuations $v : A_{\mathrm{inf}} \to \Gamma_v \cup \{0\}$ such that $v(f) \leq 1$ for all $f \in A_{\mathrm{inf}}$, where Γ_v is a totally ordered abelian group with $0 < \gamma$ for all $\gamma \in \Gamma_v$. Here, continuity means, for all $\gamma \in \Gamma_v$, $\{f \in A_{\mathrm{inf}} \mid v(f) \leq \gamma\} \subset A_{\mathrm{inf}}$ is open (for the $(p, [\varpi])$-adic topology on A_{inf}).
- The set $V(p[\varpi])$ is $\{v \in \mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}}) \mid v \text{ factors through } A_{\mathrm{inf}}/p[\varpi]\}$.
- We want to make Y_F into a locally ringed space as follows. Let
$$B^b = A_{\mathrm{inf}}[\frac{1}{p[\varpi]}] = \left\{ \sum_{n \gg -\infty} [x_n]p^n \in W(F)[1/p] \mid x_n \text{ is bounded} \right\}.$$
This should be thought of as (bounded) functions on Y_F. For $0 \leq \rho \leq 1$, define a norm $|-|_\rho$ on B^b as
$$\left| \sum_{n \gg -\infty} [x_n]p^n \right|_\rho := \max_n |x_n| \rho^n \in \mathbb{R}_{\geq 0}.$$
For a finite union of closed intervals $I \subset [0,1]$, define $B_I = B_{F,I}$ to be the completion of B^b with rrespect to the family of norms $\{|-|_\rho\}_{\rho \in I}$.

  Theorem 6.1 (Fargues-Fontaine)**.** *If $1 \notin I$, and if the endpoints of I are in $|F^\times|$, then B_I is a PID.*

  Definition 6.3. *We define*
  $$Y_{F,I} = \mathrm{Spa}(B_I, B_I^\circ) = \{v \in Y_F \text{ that extends to a continuous valuation on } B_I\}.$$

  Then, $Y_{F,I_1} \cap Y_{F,I_2} = Y_{F,I_1 \cap I_2}$ and $Y_F = \bigcup_{I \subset (0,1)} Y_{F,I}$. The topology on Y_F can be defined as the topology generated by regarding $Y_{F,I}$ as open subsets of Y_F. Furthermore, one can equip Y_F with a structure sheaf $\mathcal{O}_{Y_F}$ such that $\Gamma(Y_{F,I}, \mathcal{O}_{Y_F}) = B_I$, which makes Y_F into a locally ringed space.
- For $f \in B^b$, $|\varphi(f)|_{\rho^p} = |f|_\rho^p$, so φ extends to $B_{[a,b]} \xrightarrow{\sim} B_{[a^p,b^p]}$, so $\varphi : Y_{F,[a^p,b^p]} \xrightarrow{\sim} Y_{F,[a,b]}$, and $\varphi : Y_F \xrightarrow{\sim} Y_F$, giving a totally disconnected and free action of $\mathbb{Z}$.

Now we have a locally ringed space with totally disconnected free action by $\mathbb{Z}$, the following makes sense.

Definition 6.4. *The (adic)* ***Fargues-Fontaine curve*** *is the locally ringed space $X_F = Y_F/\varphi^{\mathbb{Z}}$.*

Remark 6.1. If $F = \overline{F}$, then there is a bijection

$$\{\text{“classical points” of } Y_F\} = \{(p - [a]) \subset A_{\mathrm{inf}}, a \in \mathcal{O}_F, 0 < |a| < 1\},$$

where a classical point means a point $v \in \mathrm{Spa}(B_I, B_I^\circ)$ for some nice I which factors through a maximal ideal of B_I (this comes from the PID-ness of B_I's). Then, there is a bijection

$$\{(p - [a]) \subset A_{\mathrm{inf}}, a \in \mathcal{O}_F, 0 < |a| < 1\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{perfectoid fields } E \\ \text{of characteristic } 0, \\ E^\flat \cong F \end{array} \right\},$$

$$(p - [a]) \mapsto W(\mathcal{O}_F)/(p - [a]).$$

In a certain senes, Y_F is a "punctured open unit disc," with the coordinate function being p.

Let $B = \Gamma(Y_F, \mathcal{O}_{Y_F}) = \varprojlim_I B_I$.

Theorem 6.2 (Fargues-Fontaine). *We have* $\Gamma(X_F, \mathcal{O}_{X_F}) = B^{\varphi=\mathrm{id}} = \mathbb{Q}_p$.

Now we want to make sense of the notion of "vector bundles over X_F". Note that, at least formally, a vector bundle over X_F must be a vector bundle V on Y_F with $\varphi^* V \xrightarrow{\sim} V$.

Theorem 6.3. *The embedding*

$$\left\{ \begin{array}{c} \varphi\text{-module over } B, \\ \text{finite projective over} \\ B \end{array} \right\} \hookrightarrow \left\{ \begin{array}{c} \text{vector bundle } V \text{ on} \\ Y_F \text{ with } \varphi^* V \xrightarrow{\sim} V \end{array} \right\},$$

is a bijection. That is, a vector bundle V on Y_F is finite projective over B.

Remark 6.2. This is not an equivalence of categories, in particular this is not full (the Hom space on the vector bundle side is huge).

An advantage of this approach is that we can use geometric intuition.

Proposition 6.1. *Any closed line bundle $\mathscr{L}$ on X_F is associated to a divisor $D = \sum_{i<\infty} n_i x_i$, $n_i \in \mathbb{Z}$, and $x_i \in X_F$ classical points. For a classical point x, $\mathcal{O}(-x)$ is the ideal sheaf of functions vanishing at x.*

Definition 6.5. *If $\mathscr{L} \cong (\sum n_i x_i)$ is a line bundle, let* $\deg \mathscr{L} = \sum n_i$. *If V is a vector bundle on X_F, define* $\deg V = \deg(\Lambda^{\mathrm{rk}\, V} V)$, *and* $\mu(V) = \frac{\deg(V)}{\mathrm{rk}(V)}$, *the* ***slope*** *of V.*

Definition 6.6. *A vector bundle V is called* ***semistable of slope*** *μ if $\mu(V) = \mu$, and for any subbundle $V' \subset V$, $\mu(V') \leq \mu(V)$.*

Theorem 6.4 (Fargues-Fontaine).
1. *Every vector bundle on X_F decomposes as a direct sum of semistable vector bundles.*
2. *For every $\mu \in \mathbb{Q}$, there is a unique indecomposable semistable vector bundle V_μ of slope μ.*

We review the construction of V_μ.

Definition 6.7. *Let $\mu = \frac{r}{s}$ where $r, s \in \mathbb{Z}$, $s \geq 1$, $(r, s) = 1$. Let*

$$D_\mu = (W(\overline{\mathbb{F}}_p)[1/p])^s.$$

This can be made into a φ-module over $W(\overline{\mathbb{F}}_p)[1/p]$ by giving

$$\varphi = \begin{pmatrix} 0 & \cdots & \cdots & 0 & p^r \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & 1 & 0 \end{pmatrix},$$

with respect to the canonical basis. Let $\mathcal{D}_\mu$ be the pullback (as a φ-module) of D_μ into $A_{\mathrm{inf}}[\frac{1}{p[\varpi]}]$ along

$$\begin{array}{ccc} A_{\mathrm{inf}} & \longleftarrow & W(\overline{\mathbb{F}}_p) \\ \downarrow & & \downarrow \\ \mathcal{O}_F & \longleftarrow & \overline{\mathbb{F}}_p \end{array}$$

This defines a vector bundle $\mathcal{D}_\mu$ on Y_F with $\varphi^*\mathcal{D}_\mu \xrightarrow{\sim} \mathcal{D}_\mu$, and this descends to a vector bundle V_μ on X_F.

Definition 6.8 (Rational subdomains)**.** *If $f_1, \cdots, f_n, g \in B_I$, $(f_1, \cdots, f_n, g) = (1)$, we deinfe*

$$U\left(\frac{f_1, \cdots, f_n}{g}\right) := \{x \in Y_{F,I} \mid |f_i(x)| \leq |g(x)|\},$$

where $x = v \in Y_{F,I}$ is a valuation, $|f(x)| = v(f)$.

One could easily check that if $I' \subset I$ is a subinterval with endpoints in $|F^\times|$, $Y_{F,I'} \subset Y_{F,I}$ is of this form.

Example 6.1. For an algebraically closed field F, a classical point $x \in Y_F$ is of form $(p - [a])$ for $a \in \mathcal{O}_F$ with $0 < |a| < 1$. Then, $\mathcal{O}_E := A_{\mathrm{inf}}/(p - [a])$ defines a perfectoid field $E = \mathcal{O}_E[1/p]$ where $E^\flat \cong F$ via $\mathcal{O}_E/p \cong \mathcal{O}_F/a$. For $f \in A_{\mathrm{inf}}$, let $\overline{f}$ be the image of f in $\mathcal{O}_E/p = \mathcal{O}_F/a$. If $\overline{f} \neq 0$, then for a lift $\overline{f}' \in \mathcal{O}_F$, we have $|f(x)| = |\overline{f}'|$, which should be independent of the lift.

The classification of vector bundles on X_F, for F algebraically closed, implies

Corollary 6.1. *There is a bijection*

$$\left\{\begin{array}{c}\textit{semistable vector}\\ \textit{bundles on } X_F \textit{ of}\\ \textit{slope 0}\end{array}\right\} \xrightarrow{\sim} \left\{\textit{finite dimensional } \mathbb{Q}_p\textit{-vector spaces}\right\},$$

$$V \mapsto \Gamma(X_F, V),$$
$$V \otimes_{\mathbb{Q}_p} \mathcal{O}_{X_F} \leftarrow V.$$

Remark 6.3. In general you should not expect $\Gamma(X_F, V)$ to be finite-dimensional.

Remark 6.4. For an arbitrary F, the curve X_F has an algebraic variant

$$X_F^{\mathrm{alg}} = \mathrm{Proj} \bigoplus_{d \geq 0} B^{\varphi = p^d}.$$

This enjoys a strong finiteness property.

Theorem 6.5 (Fargues-Fontaine)**.** *(1) X_F^{alg} is a regular 1-dimensional noetherian scheme.*

(2) There is a morphism of locally ringed spaces $X_F \longrightarrow X_F^{\mathrm{alg}}$ which induces a bijection

$$\{\textit{classical points of } X_F\} \xrightarrow{\sim} \left\{\textit{closed points of } X_F^{\mathrm{alg}}\right\},$$

which further induces a bijection of completed local rings at classical points.

Theorem 6.6 (GAGA; Kedlaya-Liu)**.** *For F algebraically closed, pullback along $X_F \longrightarrow X_F^{\mathrm{alg}}$ gives an equivalence*

$$\left\{\textit{vector bundles on } X_F^{\mathrm{alg}}\right\} \xrightarrow{\sim} \{\textit{vector bundles on } X_F\}.$$

Remark 6.5. For F not algebraically closed, this may not hold.

The reason we might be interested in schematic Fargues-Fontaine curve is because we can give a vector bundle on it by giving a vector bundle on an open subset, the complement of a point, and a modification datum at that point. This does not work in the world of adic spaces, because functions can have essential singularity around a point.

Definition 6.9. *Let $\infty \in X_F^{\mathrm{alg}}$ be any closed point. Define*

$$B_e = \Gamma(X_F^{\mathrm{alg}} \backslash \{\infty\}, \mathcal{O}_{X_F^{\mathrm{alg}}}) = B[1/t]^{\varphi=\mathrm{id}},$$

for some $t \in B$ such that $V(t) = \mathrm{pr}^{-1}(\infty) \subset Y_F$.

Thus, we have an equivalence

$$\left\{ \text{vector bundles on } X_F^{\mathrm{alg}} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{finite projective} \\ B_e\text{-module } M \text{ with} \\ \widehat{\mathcal{O}}_{X_F^{\mathrm{alg}},\infty}\text{-lattice} \\ \Lambda \subset M \otimes_{B_e} \widehat{\mathcal{O}}_{X_F^{\mathrm{alg}},\infty}[1/t] \end{array} \right\},$$

where $t \in \widehat{\mathcal{O}}_{X_F^{\mathrm{alg}},\infty}$ is a pseudouniformizer.

7. Equivariant vector bundles.

Let $K/\mathbb{Q}_p$ be a finite extension, and $\mathbb{C}_p = \widehat{\overline{K}}$ which has an action by G_K. Then, G_K acts on $\mathbb{C}_p^\flat$ and on $W(\mathcal{O}_{\mathbb{C}_p^\flat}) = A_{\mathrm{inf}}(\mathbb{C}_p^\flat)$, thus on $Y_{\mathbb{C}_p^\flat}$ and $X_{\mathbb{C}_p^\flat}$.

Corollary 7.1. *There is a bijection,*

$$\left\{ \begin{array}{c} \text{continuous} \\ \text{representations of} \\ G_K \text{ on finite} \\ \text{dimensional} \\ \mathbb{Q}_p\text{-vector spaces} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} G_K\text{-equivariant} \\ \text{vector bundle on} \\ X_{\mathbb{C}_p^\flat}, \text{ semistable of} \\ \text{slope } 0 \end{array} \right\},$$

$$V \mapsto V \otimes_{\mathbb{C}_p} \mathcal{O}_{X_{\mathbb{C}_p^\flat}},$$

where G_K acts diagonally on the RHS.

Remark 7.1. If V is an equivariant vector bundle, $U \subset X_{\mathbb{C}_p^\flat}$ open and $H \subset G_K$ is the stabilizer of U, then H is asked to act continuously on $\Gamma(U, V)$.

Corollary 7.2. *Let $\theta : W(\mathcal{O}_{\mathbb{C}_p^\flat}) \to \mathcal{O}_{\mathbb{C}_p}$ be the continuous surjective map which defines a classical point $x_0 \in Y_{\mathbb{C}_p^\flat}$. Let $\infty \in X_{\mathbb{C}_p^\flat}$ be its image. Then, ∞ is stabilized by G_K, and this endows a G_K-action on $B_e = \Gamma(X_{\mathbb{C}_p^\flat}^{\mathrm{alg}} \backslash \{\infty\}, \mathcal{O}_{X_{\mathbb{C}_p^\flat}^{\mathrm{alg}}})$. Then,*

$$\left\{ \begin{array}{c} G_K\text{-equivariant} \\ \text{vector bundles on} \\ X_{\mathbb{C}_p^\flat} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} (M, \Lambda) \text{ where } M \text{ is finite projective} \\ B_e\text{-module with semilinear } G_K\text{-action,} \\ \Lambda \subset M \otimes_{B_e} \widehat{\mathcal{O}}_{X_{\mathbb{C}_p^\flat},\infty}[1/t] \text{ is a } G_K\text{-stable} \\ \widehat{\mathcal{O}}_{X_{\mathbb{C}_p^\flat},\infty}\text{-lattice} \end{array} \right\},$$

is a bijection.

Definition 7.1. *We write*

$$B_{\mathrm{dR}}^{+} = \widehat{\mathcal{O}}_{X_{\mathbb{C}_p^\flat},\infty} = \widehat{\mathcal{O}}_{Y_{\mathbb{C}_p^\flat},x_0},$$

which is the completion of $W(\mathcal{O}_{\mathbb{C}_p^\flat})[1/p]$ *with respect to* $\ker\theta$.

Example 7.1. Let us denote $\underline{\varepsilon} = (\varepsilon_n, n \geq 0) \in \mathcal{O}_{\mathbb{C}_p^\flat}$, and let $t = \log([\underline{\varepsilon}])$, which means

$$t = \sum_{n\geq 1} (-1)^{n+1} \frac{([\underline{\varepsilon}]-1)^n}{n},$$

which does not converge in A_{inf} or B^b. However, it converges in all B_I with $I \subset (0,1)$, so it converges in B. In particular, t is a uniformizer in $\widehat{\mathcal{O}}_{Y_{\mathbb{C}_p^\flat},x_0}$, $\varphi(t) = pt$, and for $g \in G_K$, $gt = \chi(g)t$. Then, $V(t) = \mathrm{pr}^{-1}(\infty) \subset Y_F$.

Consider the following table, which summarizes corresponding objects in different settings.

Equivariant v.b. on $X_{\mathbb{C}_p^\flat}$	φ-module on B	φ-module on $W(\overline{\mathbb{F}}_p)[1/p] = \breve{\mathbb{Q}}_p$
$\mathcal{O}_{X_{\mathbb{C}_p^\flat}}$	$(B, \varphi(1) = 1)$	$D_0 = (\breve{\mathbb{Q}}_p, \varphi(1) = 1)$
$\mathcal{O}_{X_{\mathbb{C}_p^\flat}}(\infty)$	$t^{-1}B$ ($\cong (B, \varphi(1) = p^{-1})$)	$D_{-1} = (\breve{\mathbb{Q}}_p, \varphi(1) = p^{-1})$

In particular, as $\mathcal{O}_{X_{\mathbb{C}_p^\flat}}$ is semistable of slpoe zero, we get a trivial G_K-representaiton on $\mathbb{Q}_p$, whereas as $\mathcal{O}_{X_{\mathbb{C}_p^\flat}}(\infty)$ is semistable of slope 1, $\Gamma(X_{\mathbb{C}_p^\flat}, \mathcal{O}_{X_{\mathbb{C}_p^\flat}}(\infty)) = (t^{-1}B)^{\varphi=\mathrm{id}} = B^{\varphi=p}$ is infinite-dimensional. In particular, $\mathrm{Hom}(D_0, D_{-1}) = 0$, but $\mathrm{Hom}(\mathcal{O}_{X_{\mathbb{C}_p^\flat}}, \mathcal{O}_{X_{\mathbb{C}_p^\flat}}(\infty)) = B^{\varphi=p}$ is huge.

On the other hand, if we consider the line bundle $\mathscr{L}$ on $X_{\mathbb{C}_p^\flat}$ which corresponds to a φ-module over B, $(t^{-1}B, \phi = p\varphi)$, then $\mathscr{L}$ is semistable of slope zero and the corresponding Galois representation is $t^{-1}\mathbb{Q}_p = (t^{-1}B)^{\phi=\mathrm{id}}$, with G_K acting via χ^{-1}.

8. **Galois descent, decompletion and deperfection.**

Let F be any perfectoid field of characteristic p. Let $C = \widehat{\overline{F}} = \widehat{F}^s$ which has an action of G_F. Then, $F \longrightarrow C$ induces $X_C \longrightarrow X_F$, $X_C^{\mathrm{alg}} \longrightarrow X_F^{\mathrm{alg}}$. These morphisms are equivariant for G_F-action.

Theorem 8.1 ((Pro-)Galois descent)**.** *The natural map*

$$\left\{ \textit{vector bundles on } X_F^{\mathrm{alg}} \right\} \longrightarrow \left\{ \begin{array}{c} G_F\textit{-equivariant} \\ \textit{vector bundle on} \\ X_C^{\mathrm{alg}} \end{array} \right\},$$

is an equivalence of categories.

Now let $K/\mathbb{Q}_p$ be a finite extension, K_∞/K be a Galois extension, $\Gamma = \mathrm{Gal}(K_\infty/K)$ such that $\widehat{K}_\infty$ is perfectoid.

Corollary 8.1. *There is a natural equivalence of categories*

$$\left\{ \begin{array}{c} \Gamma\textit{-equivariant} \\ \textit{vector bundles on} \\ X_{\widehat{K}_\infty^\flat}^{\mathrm{alg}} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} G_K\textit{-equivariant} \\ \textit{vector bundles on} \\ X_{\mathbb{C}_p^\flat}^{\mathrm{alg}} \end{array} \right\}.$$

Remark 8.1. We have an equivalence

$$\left\{ \begin{array}{c} \text{vector bundles on} \\ X_{\widehat{K}_\infty^\flat} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \varphi\text{-modules over} \\ B = \Gamma(Y_{\widehat{K}_\infty^\flat}, \mathcal{O}_{Y_{\widehat{K}_\infty^\flat}}) \end{array} \right\}.$$

By restriction, one sees that this is equivalent to

$$\xrightarrow{\sim}\left\{\begin{array}{c}\varphi\text{-modules over}\\ B^{(0,r]}\end{array}\right\}=\left\{\begin{array}{c}B^{(0,r]}\text{-module } M,\\ \text{with } \varphi^*M\xrightarrow{\sim}\\ M\otimes_{B^{(0,r]}}B^{(0,r^p]}\end{array}\right\}\xrightarrow{\sim}\left\{\begin{array}{c}\varphi\text{-module over}\\ \lim_{r\to 0}B^{(0,r]}=:\\ \widetilde{B}^{\dagger}_{\mathrm{rig},K}\end{array}\right\}.$$

A similar statement holds for (φ,Γ)-modules.

Let Δ^* be the punctured open unit disc over $W(\mathbb{F}_q)[1/p]$, where $\mathbb{F}_q$ is residue field of K_∞. Then $\Gamma(\Delta^*,\mathcal{O}_{\Delta^*})\hookrightarrow B$; if $K/\mathbb{Q}_p$ is unramified, then one can explicitly describe this as $X\mapsto[\underline{\varepsilon}]-1$ where X is the coordinate function on Δ^*. Again, each finite level annulus is not stable under φ and Γ, but the limit of these towards boundary is:

$$\mathcal{R}=\lim_{\rho\to 1^-}\Gamma(\{\rho\le|x|<1\},\mathcal{O}_{\Delta^*})\subset\widetilde{B}^{\dagger}_{\mathrm{rig},K},$$

is stable under φ,Γ (although φ is not injective). This is usually referred as the **Robba ring** (or $B^{\dagger}_{\mathrm{rig},K}$ in Berger's terminology). This p-adic limit process gives a compactification by adding characteristic p points (which is why adic space is useful).

Now let K_∞ be the cyclotomic extension. Then, much more can be said, which is specific to cyclotomic extension.

Theorem 8.2 (Decompletion and deperfection)**.**

$$\{(\varphi,\Gamma)\textit{-modules over }\mathcal{R}\}\xrightarrow{\sim}\left\{(\varphi,\Gamma)\textit{-modules over }\widetilde{B}^{\dagger}_{\mathrm{rig},K}\right\},$$

is a bijection.

This is really only specific to the cyclotomic extension.

9. **Crystalline representations and Fontaine's period rings.**

How do we produce G_K-equivariant vector bundles on $X_{\mathbb{C}_p^\flat}$, semistable of slope zero? We saw that a φ-module over $\breve{\mathbb{Q}}_p$ produces a vector bundle on $X_{\mathbb{C}_p^\flat}$. Thus, a G_K-equivariant φ-module over $\breve{\mathbb{Q}}_p$ will produce a G_K-equivariant vector bundle on X. As $\breve{\mathbb{Q}}_p^{G_K}=W(k)[1/p]=K_0$, a φ-module over K_0 will produce a G_K-equivariant vector bundle on $X_{\mathbb{C}_p^\flat}$. Let's call this functor

$$(D,\varphi)\mapsto V(D,\varphi).$$

Of course, there is no guarantee that $V(D,\phi)$ is semistable of slope zero. We can instead try to modify the vector bundle at ∞ and hope to get something semistable of slope zero. Certainly, for a $B^+_{\mathrm{dR}}=\widehat{\mathcal{O}}_{X_{\mathbb{C}_p^\flat},\infty}$-lattice Λ in $V(D,\varphi)\otimes\widehat{\mathcal{O}}_{X_{\mathbb{C}_p^\flat},\infty}[1/t]$, we can modify $V(D,\varphi)$ by Λ, and get a vector bundle $V(D,\varphi,\Lambda)$. As G_K acts on $V(D,\varphi)\otimes\widehat{\mathcal{O}}_{X_{\mathbb{C}_p^\flat},\infty}[1/t]$, if Λ is a G_K-stable lattice, then $V(D,\varphi,\Lambda)$ is a G_K-equivariant vector bundle. If $V(D,\varphi,\Lambda)$ is semistable of slope zero, then $\Gamma(X,V(D,\varphi,\Lambda))$ is a continuous G_K-representation on a finite dimensional $\mathbb{Q}_p$-vector space, with dimension $\mathrm{rk}\,V(D,\varphi)=\dim_{K_0}D$.

Crystalline representations are precisely the G_K-representations which can arise in this way.

Definition 9.1. *Let A_{cris} be the p-adic completion of the divided power envelope of $A_{\mathrm{inf}}=A_{\mathrm{inf}}(\mathcal{O}_{\mathbb{C}_p^\flat})$ with respect to* $\ker(A_{\mathrm{inf}}\xrightarrow{\theta}\mathcal{O}_{\mathbb{C}_p})$*. Namely,*

$$A_{\mathrm{cris}}=\left(A_{\mathrm{inf}}\left[\frac{\xi^n}{n!},n\ge 1\right]\right)^{\wedge},$$

where ξ is a generator of the principal ideal $\ker\theta$.

Let $B^+_{\mathrm{cris}} = A_{\mathrm{cris}}[1/p]$, and $B_{\mathrm{cris}} = B^+_{\mathrm{cris}}[1/t] = A_{\mathrm{cris}}[1/t]$, where $t = \log[\underline{\varepsilon}]$.

By construction, one can easily check the following

Proposition 9.1.

- *The G_K-action on A_{inf} extends to A_{cris}, B^+_{cris} and B_{cris}.*
- *The Frobenius $\varphi : A_{\mathrm{inf}} \xrightarrow{\sim} A_{\mathrm{inf}}$ extends to $\varphi : A_{\mathrm{cris}} \hookrightarrow A_{\mathrm{cris}}$ and $\varphi : B^{(+)}_{\mathrm{cris}} \longrightarrow B^{(+)}_{\mathrm{cris}}$, commuting with G_K-action.*
- *There is a natural G_K-equivariant injective ring homomorphism $B^+_{\mathrm{cris}} \hookrightarrow B^+_{\mathrm{dR}}$.*

Remark 9.1. The injection $B^+_{\mathrm{cris}} \hookrightarrow B^+_{\mathrm{dR}}$ is justified by showing that an expansion in B^+_{cris} converges in B^+_{dR}. This is a subtle issue, as the topology of B^+_{cris} is p-adic, whereas the topology of B^+_{dR} is t-adic (or valuation topology). Thus, the injective ring homomorphism is not really compatible with topology.

Definition 9.2. *The $\mathbb{Z}$-graded filtration $\mathrm{Fil}^i B_{\mathrm{cris}}$ is defined by $\mathrm{Fil}^i B_{\mathrm{cris}} = B_{\mathrm{cris}} \cap t^i B^+_{\mathrm{dR}}$.*

Remark 9.2. Even though t makes sense in B^+_{cris}, $\mathrm{Fil}^i B_{\mathrm{cris}}$ is not the same as $t^i B^+_{\mathrm{cris}}$.

Proposition 9.2. *The rings B_{cris} and B_{dR} are G_K-regular. Also,*

- $B^{G_K}_{\mathrm{cris}} = K_0$,
- $B^{G_K}_{\mathrm{dR}} = K$,
- $(B^+_{\mathrm{cris}})^{\varphi=\mathrm{id}} = bQ_p$.

Definition 9.3. *A continuous G_K-representation V over a finite dimensional $\mathbb{Q}_p$-vector space is called* ***de Rham*** *if V is B_{dR}-admissible, and* ***crystalline*** *if V is B_{cris}-admissible. We define functors*

$$D_{\mathrm{dR}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_K},$$

which is a finite-dimensional K-vector space, and

$$D_{\mathrm{cris}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{G_K},$$

which is a finite-dimensional K_0-vector space.

Remark 9.3. As $B_{\mathrm{cris}} \hookrightarrow B_{\mathrm{dR}}$ Galois-equivariantly, crystalline representations are automatically de Rham. In this case,

$$D_{\mathrm{cris}}(V)\otimes_{K_0} K = (D_{\mathrm{cris}}(V)\otimes_{K_0} B_{\mathrm{dR}})^{G_K} = (D_{\mathrm{cris}}(V)\otimes_{K_0} B_{\mathrm{cris}}\otimes_{B_{\mathrm{cris}}} B_{\mathrm{dR}})^{G_K} = (V\otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}\otimes_{B_{\mathrm{cris}}} B_{\mathrm{dR}})^{G_K} = D_{\mathrm{dR}}(V).$$

But $D_{\mathrm{cris}}(V)$ and $D_{\mathrm{dR}}(V)$ contain more structures. As $\varphi : B_{\mathrm{cris}} \longrightarrow B_{\mathrm{cris}}$ and $t^i B^+_{\mathrm{dR}} \subset B_{\mathrm{dR}}$ are compatible with G_K-action, we have

$$\phi : D_{\mathrm{cris}}(V) \longrightarrow D_{\mathrm{cris}}(V),$$

which is φ-linear bijection, and a $\mathbb{Z}$-filtration of K-vector spaces

$$\mathrm{Fil}^i D_{\mathrm{dR}}(V) = (V \otimes_{\mathbb{Q}_p} t^i B^+_{\mathrm{dR}})^{G_K},$$

which is exhaustive and separated. Based on this, we have the following definition.

Definition 9.4. *A* ***filtered φ-module*** *for K is a finite dimensional K_0-vector space D with a φ-linear isomorphism $\phi : D \longrightarrow D$ and an exhaustive separated filtartion $\mathrm{Fil}^i D_K \subset D_K = D \otimes_{K_0} K$ of sub-K-vector spaces.*

We have seen that D_{cris} gives a functor

$$D_{\mathrm{cris}} : \{\text{crystalline representations}\} \longrightarrow \{\text{filtered } \varphi\text{-modules}\}.$$

Theorem 9.1 (Fontaine, Colmez-Fontaine, Berger, Kedlaya, Kisin). *(1) D_{cris} is fully faithful.*

*(2) The essential image of D_{cris} is given by the **weakly admissible** objects, and a quasi-inverse to D_{cris} is given by*

$$V_{\mathrm{cris}}(D, \phi, \mathrm{Fil}^\bullet) := \mathrm{Fil}^0((D \otimes_{K_0} B_{\mathrm{cris}})^{\varphi=\mathrm{id}}) = (D \otimes_{K_0} B_{\mathrm{cris}})^{\varphi=\mathrm{id}} \cap \mathrm{Fil}^0(D_K \otimes_K B_{\mathrm{dR}}) \subset D \otimes_{K_0} B_{\mathrm{dR}} = D_K \otimes_K B_{\mathrm{dR}}.$$

Definition 9.5. *A filtered φ-module $(D, \phi, \mathrm{Fil}^\bullet)$ is **weakly admissible** if it is semistable of slope zero for the slope theory defined by*

$$\mu(D, \phi, \mathrm{Fil}^\bullet) = v_p(\det \phi) - \sum_{i \in \mathbb{Z}} i \dim_K \mathrm{gr}_i D_K.$$

Remark 9.4. Let X/K be a proper smooth algebraic variety. Then, $V = H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is a finite-dimensional continuous G_K-representation which is always de Rham. There is a canonical G_K-equivariant isomorphism

$$H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \xrightarrow{\sim} H^i_{\mathrm{dR}}(X/K) \otimes_K B_{\mathrm{dR}}.$$

This gives a canonical identification $D_{\mathrm{dR}}(V) = H^i_{\mathrm{dR}}(X/K)$. This in fact identifies the filtration on $D_{\mathrm{dR}}(V)$ and the Hodge filtration on $H^i_{\mathrm{dR}}(X/K)$.

If X furthermore has good reduction so that there is a smooth proper model $\mathcal{X}/\mathcal{O}_K$, then $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is crystalline, and there is a canonical G_K and φ-equivariant isomorphism

$$H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}} \cong H^i_{\mathrm{cris}}(\mathcal{X}_k/W(k)) \otimes_{W(k)} \otimes B_{\mathrm{cris}},$$

so that $D_{\mathrm{cris}}(V) = H^i_{\mathrm{cris}}(\mathcal{X}_k/W(k))[1/p]$ as φ-modules over K_0.

We want to relate the classical theory to the geometric theory of Fargues-Fontaine curve alluded earlier.

Proposition 9.3. *One has identifications*

$$(B^+_{\mathrm{cris}}[1/t])^{\varphi=\mathrm{id}} = B^{\varphi=\mathrm{id}}_{\mathrm{cris}} = B_e = (B[1/t])^{\varphi=\mathrm{id}},$$

$$(B^+_{\mathrm{cris}})^{\varphi=p^d} = B^{\varphi=p^d}.$$

In particular,

$$X^{\mathrm{alg}}_{\mathbb{C}_p^\flat} = \mathrm{Proj} \bigoplus_d B^{\varphi=p^d} = \mathrm{Proj} \bigoplus_d (B^+_{\mathrm{cris}})^{\varphi=p^d}.$$

From this, one sees that there is a functor

$$\{\varphi\text{-module over } K_0\} \longrightarrow \{\varphi\text{-module over } B^+_{\mathrm{cris}}\} \xrightarrow{\mathrm{gr}^\bullet} \left\{ \begin{matrix} \text{graded} \\ \bigoplus_d (B^+_{\mathrm{cris}})^{\varphi=p^d}\text{-} \\ \text{module} \end{matrix} \right\} \longrightarrow \left\{ \text{vector bundle on } X^{\mathrm{alg}}_{\mathbb{C}_p^\flat} \right\},$$

and this turns out to coincide with $(D, \phi) \mapsto V(D, \phi)$ we built earlier.

Lemma 9.1. *There is a bijection*

$$\left\{ \begin{matrix} \textit{exhaustive} \\ \textit{separable filtrations} \\ \textit{of } D_K \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} G_K\textit{-stable} \\ B^+_{\mathrm{dR}}\textit{-lattice} \\ \Lambda \subset V(D, \varphi) \otimes B_{\mathrm{dR}} = \\ D_K \otimes_K B_{\mathrm{dR}} \end{matrix} \right\},$$

$$(\mathrm{Fil}^i D_K)_i \mapsto \sum \mathrm{Fil}^i D_K \otimes t^{-i} B^+_{\mathrm{dR}}.$$

We now compute what $V(D,\phi,\Lambda)$ is, for Λ coming from the filtration of a filtered φ-module $(D,\phi,\mathrm{Fil}^\bullet)$:

$$\begin{aligned}\Gamma(X^{\mathrm{alg}}_{\mathbb{C}_p^\flat}, V(D,\phi,\Lambda)) &= \Gamma(X^{\mathrm{alg}}\backslash\{\infty\}, V(D,\phi)) \cap \Lambda \\ &= (D \otimes_{K_0} B_{\mathrm{cris}})^{\varphi=\mathrm{id}} \cap \Lambda \\ &= V_{\mathrm{cris}}(D,\phi,\mathrm{Fil}^\bullet).\end{aligned}$$

We see that the slope theory of filtered φ-modules coincides with the slope theory of vector bundles over algebraic Fargues-Fontaine curve, and thus we get the following

Proposition 9.4. *The vector bundle $V(D,\phi,\Lambda)$ is semistable of slope zero if and only if $(D,\phi,\mathrm{Fil}^\bullet)$ is weakly admissible.*

Remark 9.5. If one fills out the details of the proof of this, this will actually give the full proof of Theorem 9.1.