THE MONODROMY-WEIGHT CONJECTURE

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Deligne [D1] formulated his conjecture in 1970, simultaneously in the ℓ-adic and Hodge theoretic settings. The Hodge theoretic statement, amounted to the existence of what is now called a limit mixed Hodge structure. This was solved by Schmid [S] a couple of years later. I won't elaborate, since it would be too much of a detour. I will explain the ℓ-adic version and Deligne's solution [D3] in an important special case. Ito [I] proved the conjecture in equicharacteristic by reducing to Deligne's result. As I understand it, Scholze [Sc] reduces his result in mixed characteristic to this case as well.

It's easier (at least for me) to start with the complex picture. Let Δ be a disk, and suppose that $X \subset \mathbb{P}^d_{\mathbb{C}} \times \Delta$ is a submanifold such that projection $f : X \to \Delta$ is onto. Let $X_t = f^{-1}(t)$, $\Delta^* = \Delta - \{0\}$ and $f^* : X - X_0 \to \Delta^*$ be the restriction. Then after shrinking Δ, we can assume that f^* is smooth. Therefore, by a theorem of Ehresmann, f^* is topologically a fibre bundle, i.e. it is locally a product of Δ^* with a space $F \cong X_t$. To understand the topology more clearly, let us restrict to a circle $S^1 \subset \Delta^*$. S^1 is gotten by gluing 0 to 1 in the interval $[0, 1]$. Likewise the bundle is given by gluing the ends $F \times \{0\}$ to $F \times \{1\}$ by a self homeomorphism $h : F \to F$. Although this construction involved many choices, the action

$$T = h^* : H^i(F, \mathbb{Q}) \to H^i(F, \mathbb{Q})$$

is independent of them. T is called *monodromy*. This defines a representation

$$\mathbb{Z} = \pi_1(\Delta^*, t) \to Aut(H^i(X_t, \mathbb{Q})), \quad n \mapsto T^n$$

In the topological setting, T could be almost anything, but in the present setting of a family of projective manifolds there is a very strong restriction.

Theorem 1 (Borel, Grothendieck, Landman)**.** *T is a quasi-unipotent matrix, i.e. the eigenvalues of T are roots of unity.*

We indicate Grothendieck's proof since it seems the most relevant here. First, we need to make a switch to a more algebraic picture. We replace Δ with the spectrum S of Henselian[1] discrete valuation ring R. Let $k = R/m$ be the residue field, and K the fraction field. We replace $\pi_1(\Delta)$ by the inertia group I. Recall that this is determined by the exact sequence

$$1 \to I \to G_K \to G_k \to 1 \tag{1}$$

where $G_K = Gal(\bar{K}/K)$ etc, where for now $\bar{K}$ is the separable closure. If $p = char\, k$, then the prime to p part of I has a single (topological) generator like $\pi_1(\Delta^*)$. Suppose now that $f : X \to S$ is a projective scheme, by the magic of étale cohomology[2] G_K will act on the ℓ-adic cohomology of the geometric generic fibre $H^i_{et}(X_{\bar{K}}, \mathbb{Z}_\ell)$.

[1] You can substitute "complete" if that's easier.

[2] A low tech introduction is available on my webpage at http://www.math.purdue.edu/~dvb/preprints/etale.pdf

The restriction ρ of this action to I is the analogue of the monodromy representation. Grothendieck proved the following theorem (which implies the previous theorem, but I won't go into the implication).

Theorem 2 (Grothendieck)**.** *Assume that k is finite, then there is an open subgroup of $J \subset I$ such that $\rho(g)$ is unipotent for all $g \in J$.*

Proof. I will outline the argument, and refer to [ST, appendix] for more details. After passing to an open subgroup J, we can assume that ρ factors through the maximal pro-ℓ quotient I_ℓ of I. The advantage of I_ℓ is that we know what it is, namely

$$I_\ell \cong \mathbb{Z}_\ell(1) = \varprojlim \mu_{\ell^n} \tag{2}$$

We choose an element $T \in I_\ell$, which necessarily generates it topologically, and let $N = \log T$ be the logarithm (the series converges in the ℓ-adic topology). Let $K_\ell \subset K$ be the subfield generated by ℓ^nth roots of the uniformizer of R. Then we have an extension

$$1 \to I_\ell \to Gal(K_\ell/K) \to G_k \to 1$$

similar to (1). G_k acts on I by conjugation through this sequence, and it coincides with the one given by (2). In other words, G_k acts on I_ℓ through the cyclotomic character χ. We note that when applied to the Frobenius $\chi(Fr) = q$, where q is the cardinality of k. Let us write this action exponentially as $g \mapsto g^{\chi(h)}$ for $g \in I_\ell$ and $h \in G_k$. Because of the coincidence of the two actions, we see that g and $g^{\chi(h)}$ are conjugate elements in G_K. In particular, N and $\chi(h)N = \log(T^{\chi(h)})$ are conjugate. This forces $a_i = \chi(Fr)^i a_i = q^i a_i$, where a_i is the ith coefficient of the characteristic polynomial of $\rho(N)$. This implies that all the $a_i = 0$. Therefore $\rho(N)$ is nilpotent, and so $\rho(T)$ is unipotent. □

We make a few comments about the proof.

(1) We only really needed to keep track of the action of the subgroup $\mathbb{Z} \subset G_k \cong \hat{\mathbb{Z}}$ generated by $Fr \in G_k$. So could have replaced G_K by the preimage of $\mathbb{Z}$ which is the so called Weil group W_K. For some things, this seems essential.

(2) N determines the restriction of the representation to J. Writing $V = H^i_{et}(X, \mathbb{Q}_\ell)$ and suppressing ρ, we can see by an argument similar to the one above, that

$$NFr = qFrN \tag{3}$$

in $End(V)$. This means that $N : V \to V(-1)$ is a morphism of W_K-modules, where $V(-1)$ means that we twist the action by χ.

Before stating the conjecture, I need to recall some terminology. Fix a (highly noncanonical!) isomorphism $\iota : \bar{\mathbb{Q}}_\ell \cong \mathbb{C}$ and a prime power q. Let us say that $\lambda \in \bar{\mathbb{Q}}_\ell$ has weight n (with respect to these choices) if $|\lambda| = q^{n/2}$. A vector space equipped with an endomorphism given by some kind of Frobenius action is called pure[3] of weight n if all its eigenvalues have weight n. The point is that the nth cohomology of a smooth projective variety defined over $\mathbb{F}_q$ is pure of weight n by the Weil conjectures, i.e. Deligne's theorem [D2]. In our situation, which is more complicated, Serre and Tate [ST, p 514] asked whether the weights of $V =$

[3] Deligne calls this ι-purity in [D3]. There are a number of other variations of this concept.

$H^i_{et}(X, \mathbb{Q}_\ell)$ lie in $[0, 2i]$. Deligne refined this, by taking into account the monodromy. We can decompose V into a sum of (generalized) eigenspaces. Equation (3) implies that N will map the λ eigenspace of Fr on $H^i_{et}(X, \mathbb{Q}_\ell)$ to the $q\lambda$-eigenspace. Since N is nilpotent, we cannot expect this to be an isomorphism for all λ. However, the conjecture says that if we arrange the eigenspaces in a clever way then we should expect N and its powers to induce isomorphisms.

Lemma 1. *Given a finite dimensional vector space V with a nilpotent endomorphism N, there exists a unique increasing filtration $M_\bullet$ such that*

(a) $NM_i \subset M_{i-2}$
(b) $N^r : Gr^M_r V \cong Gr^M_{-r} V$.

The lemma is probably due to Deligne, but the thing I want to emphasize is that it is purely a result of linear algebra. When $N^2 = 0$, the filtration is simply $M_{-1} = im(N)$, $M_0 = \ker(N)$, $M_1 = V$.

Conjecture 1 (The Monodromy-Weight conjecture)**.** *Let $V = H^i_{et}(X, \mathbb{Q}_\ell)$ and N as above, then $Gr^M_r V$ is pure of weight $i + r$.*

To put this another way, this says that N^r induces an isomorphism between the $i + r$ and $i - r$ pure subquotients of V.

Theorem 3 (Deligne [D3, 1.8.4])**.** *The conjecture holds when $X \to S$ is obtained from a family of projective varieties over a curve defined over a finite field.*

Deligne uses this as a step in his proof of the generalized Weil conjectures. I'm sure it has many other number theoretic applications as well. Here is an interesting consequence in topology.

Corollary 1 (Local invariant cycle theorem)**.** *Given a family $f : X \to \Delta$ over the disk, the cohomology of the singular fibre $H^i(X_0, \mathbb{Q})$ surjects onto the monodromy invariant part of a smooth fibre $H^i(X_t, \mathbb{Q})^{\pi_1(\Delta^*)}$.*

Proof. [D3, 3.6.1] + [A] + specialization to finite fields. (This can be, and usually is, proved more directly using limit mixed Hodge structures, cf [GS].) □

I'm going to try to explain a small piece of the proof of the above theorem. Let $\mathcal{X} \to Y$ be a projective morphism of smooth separated $\mathbb{F}_q$ schemes of finite type, with $\dim Y = 1$. Let $j : U \to Y$ be an open set over which f is smooth. Let $Z = Y - U$. Choose a closed point $y \in Z$, let R be the Henselization of $\mathcal{O}_y$, $S = Spec\, R$, K its field of fractions, and $X \to S$ is the fibre product. Then $\mathcal{F} = R^i f_* \mathbb{Q}_\ell|_U$ is a lisse sheaf on U, which we extend to X by taking direct image $j_*\mathcal{F}$. In more prosaic terms, $j_*\mathcal{F}$ can be viewed as the family of cohomology spaces $H^i_{et}(X_u, \mathbb{Q}_\ell)$ for $u \in U$, together with $H^i_{et}(X_K, \mathbb{Q}_\ell)^I$ at y and similar things at other "bad" points. Each of these spaces carries an action of the Frobenius $Fr_w \in G_{k(w)}$ at $w \in Y$. By the usual Weil conjectures (i.e. Deligne's theorem [D2]) these spaces are pure of weight i when $w \in U$. Deligne calls this property pointwise purity of $\mathcal{F}$, and he formulates and proves the theorem for such sheaves. This is useful, since it allows it allows him to modify $\mathcal{F}$ as the proof proceeds. We need to understand what happens at points of Z for the extension $j_*\mathcal{F}$. We can assume without loss of generality that X is affine, so the 0th compactly supported cohomology of $j_*\mathcal{F}$

vanishes. Then the Grothendieck-Lefschetz trace formula expresses the L-function

$$\underbrace{\prod_{u\in U}\det(1-Fr_u t^{\deg u},\mathcal{F})^{-1}}_{A}\underbrace{\prod_{z\in Z}\det(1-Fr_z t^{\deg z},j_*\mathcal{F})^{-1}}_{B}=\frac{\det(1-Fr\,t,H^1_c(X,j_*\mathcal{F}))}{\det(1-Fr\,t,H^2_c(X,j_*\mathcal{F}))}$$

The information about eigenvalues at $u\in U$ can be used to show that first factor labelled A has no zeros or poles in the region $|t|<q^{-i/2-1}$, and the right side has no poles in the same region. Thus we may conclude that the second factor B on the left has no poles in this region. We can use this to bound the eigenvalues λ of Fr_z on $j_*\mathcal{F}$ for $z\in Z$ by $|\lambda|\le (q^{\deg z})^{i/2+1}$, or equivalently $\log_Q|\lambda|\le i/2+1$ where $Q=q^{\deg z}$. By applying this to the nth tensor powers of $\mathcal{F}$, he gets a similar estimate on λ^n, whence

$$\log_Q|\lambda|=\frac{1}{n}\log_Q|\lambda^n|\le\frac{i}{2}+\frac{1}{n}\to\frac{i}{2}\tag{4}$$

This provides the basic foothold. It is now remains to refine this. To explain the rest of the proof, suppose for simplicity that $N^2=0$. We reduce the case where the weights of on $\mathcal{F}_u, u\in U$ are 0 by twisting by a suitable character (when i is even we can use a power of the cyclotomic character, in general see [D3, 1.2.7]), Then we just have to check that the eigenvalues of Fr_y on Gr^M_j satisfy $|\lambda|=q^{j/2}$ for $j=-1,0,1$. In fact, it is enough to treat the cases $j=-1,0$, because $Gr^M_1\cong Gr^M_{-1}(-1)$. Then (4) gives the estimate $|\lambda|\le 1$ on the eigenvalues of Fr_y on $(j_*\mathcal{F})_y=\ker N=M_0$. This bound applies to Gr^M_0. Applying the same reasoning to the dual $\mathcal{F}^*$ gives the opposite $|\lambda|\ge 1$ on Gr^M_0, which takes care of this case. For Gr^M_{-1}, we can also apply the estimate $|\lambda|\le 1$. However, this can be dramatically improved by noting that on the square $\mathcal{F}\otimes\mathcal{F}$, $Gr^M_{-1}\otimes Gr^M_{-1}(-1)$ is a summand of Gr^M_0. Thus $|\lambda^2 q|\le 1$ or $|\lambda|\le q^{-1/2}$. Dualizing gives the opposite inequality as before.

Building on this, Ito [I] and Scholze [Sc] proved

Theorem 4 (Ito)**.** *Conjecture 1 holds when char $R=$ char R/m.*

Theorem 5 (Scholze)**.** *Conjecture 1 holds when char $R=0$ and X is a set theoretic complete intersection in a smooth projective toric variety.*

It may be worth remarking that the last condition, of being a complete intersection in a smooth projective toric variety, does impose strong restrictions. For example, by weak Lefschetz, $H^i_{et}(X_{\bar K},\mathbb{Z}_\ell)=0$ when i is odd and different from $\dim X_{\bar K}$.

Finally, I should mention that an earlier case of the conjecture, for relative dimension 2 in mixed characteristic, was done by Rapoport and Zink [RZ].

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